Frequency of Oscillations of an Error Term related to the Euler function

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Abstract

Let ϕ be the Euler function, and consider the error term H in the asymptotic formula

$$\sum_{n\leq x}\frac{\phi(n)}{n}=\frac{6}{\pi^2}x+H(x).$$

We prove that for any fixed real number A, there are at least $C_AT+O(1)$ integers $n\in[1,T]$ such that $(H(n)-A)(H(n+1)-A)<0$, where $0<C_A<1$ is a constant depending on A.

Let ϕ be the Euler function (i.e. $\phi(n)$ denotes the number of integers less than n which are relatively prime to n), and define

$$H(x) = \sum_{n \le x} \frac{\phi(n)}{n} - \frac{6}{\pi^2} x.$$

In [2], it is shown that $H(x)$ has a large number (of order T) of sign changes on integers $n \le T$. In this note, we prove that this phenomenon occurs as well for the changes in sign of $H_A(n) = H(n) - A$, where A is any fixed real number. The value $A = 3/\pi^2$ plays a special role. It is indeed known that the distribution function Δ of the values taken by H_{3/π^2} at integers is symmetric [3], whence in particular $\Delta(0) = 1/2$: so one would expect the number of changes in sign of $H_A(n)$ to be particularly important when $A = 3/\pi^2$. But the slightly surprising fact is that the only value of A for which a straightforward modification of the argument in [2] is unefficient is precisely $A = 3/\pi^2$.

Theorem *Let A be a fixed number. For all sufficiently large T, we have*

$$|\{n \in [1, T] : (H(n) - A)(H(n+1) - A) < 0\}| \ge C_A T$$

where $|\{\cdots\}|$ denotes the cardinality of the set and $0 < C_A < 1$ is a constant (depending on A).

We separate the proof into three cases: (i) $A < 3/\pi^2$, (ii) $A = 3/\pi^2$ and (iii) $A > 3/\pi^2$. Cases (i) and (iii) can be treated as in [2], §3. For case (i) replace in the argument there $D(0)$ by $D(A)$ where $D(u) = \lim_{x\to\infty} x^{-1}|\{n \le x : H(n) \le u\}|$, and note that if $H(n) < A$ and $H(m) < A$ for all integral $m \in [n, n+2h)$, then for any real $t \in [n, n+h)$ we have

$$\left| \int_t^{t+h} H(u)\, du \right| \ge (\frac{3}{\pi^2} - A)(h-2),$$

as soon as h is large enough. This comes from the fact that $H(x)$ is a straight line of slope $-6/\pi^2$ in every interval $[m, m+1)$ when m is an integer. For case (iii) consider instead the proportion $(1 - D(A))$ of integers n for which $H(n) > A$, and similarly note that if $H(n) > A$ and $H(m) > A$ for all integral $m \in [n, n+2h)$, then for any real $t \in [n, n+h)$ we have

$$\left| \int_t^{t+h} H(u)\, du \right| \ge (A - \frac{3}{\pi^2})(h-2),$$

as soon as h is large enough. It is now clear why this method doesn't work when $A = 3/\pi^2$.

From [1] and [3], we know that the distribution function $D(u)$ exists, $D(3/\pi^2) = 1/2$ and $D(u)$ is a continuous function of u. Hence, for all sufficiently large T, we have

$$|\{T \le n \le 2T : H(n) \le 3/\pi^2\}| \ge \frac{3T}{7}. \tag{1}$$

Let h be a large parameter, which will be chosen later. We divide the interval $[T, 2T]$ into divisions of length h, and group every 8 divisions to form an interval. Then the number of these newly formed intervals is $\lceil T/(8h) \rceil$, which is at most $T/(7h)$ for all sufficiently large T. For convenience, we use the symbol $\mathcal{I}$ to designate a subinterval of I consisting of the initial 6 divisions. Define

$$\mathcal{C} = \{I : H(n) \le 3/\pi^2 \text{ for some } n \in \mathcal{I}\}.$$

By (1), $|\mathcal{C}| \ge (3T/7 - (2h) \times T/(7h))/(6h) = T/(42h)$. From the continuity of $D(u)$, we can find $\epsilon > 0$ such that the set $S = \{n \le 2T : 3/\pi^2 - \epsilon \le H(n) \le 3/\pi^2\}$ has cardinality $|S| \le T/168$. Consider $J_1 = \{I \in \mathcal{C} : |I \cap S| \le h/2\}$. Then

$$\frac{h}{2}|\mathcal{C} \setminus J_1| \le \sum_{I \in \mathcal{C} \setminus J_1} |I \cap S| \le |S| \le \frac{T}{168}.$$

From this, we have $|J_1| \ge T/(100h)$. Then we can proceed with the argument in [2] on the collection J_1. Define

$$J_2 = \{I \in J_1 : H(m) \le 3/\pi^2 \text{ for all integers } m \in [n, n+h] \text{ where } n \in \mathcal{I}\}.$$

As $I \in J_2$ has at most $h/2$ elements in S and $H(m) < 3/\pi^2 - \epsilon$ if $m \notin S$, we have

$$\epsilon^2 h^3 \ll \sum_{I \in J_2} \int_n^{n+h} \left(\int_t^{t+h} H(u)\, du \right)^2 dt \le \int_T^{2T} \left(\int_t^{t+h} H(u)\, du \right)^2 dt$$

where the implied constants are independent of ϵ and h. The first inequality comes again from the fact that $H(x)$ is a straight line in every interval $[m, m+1)$ when m is an integer. But the last integral is $\ll Th$ by [2, Main Lemma]. Thus,

$$|J_1 \setminus J_2| > \frac{T}{100h} - O(\frac{T}{\epsilon^2 h^2}).$$

Our assertion follows by taking h to be a sufficiently large constant.

<u>Last Remark</u>: This method can be applied to the error term

$$E(x) = \sum_{n \le x} \frac{\sigma(n)}{n} - \frac{\pi^2}{6}x + \frac{1}{2}\log x$$

associated with the sum-of-divisors function σ as well. In this case the critical value for which the argument of case (ii) applies is $A = \pi^2/12$.

References

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